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# Extension of Massive Scalar Quasinormal Modes of Kerr and Schwarzschild Black Holes

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In the case of Kerr and Schwarzschild black holes there exist continuous standing wave solutions in the total interval $0 < r < \infty$ at certain real frequencies. This means that we can see into these black holes at these frequencies with gravitational-wave detectors, studying the radial structure inside the event horizon, too.

## 1. INTRODUCTION

A black hole can be perturbed in a variety of ways other than by incidence of gravitational waves: by an object falling into it, or by the accretion of matter surrounding it. Or, we may consider a black hole being formed by a slightly aspherical collapse of a star settling toward a final state described by the Schwarzschild or Kerr solution.[2] In all these cases, the evolution of the perturbations—if they can be considered as "small"—can, in principle, be followed by expressing them as superpositions of the basic solutions. However, we may expect on general grounds that any initial perturbation will, during its last stages, decay in a manner characteristic of the black hole and independently of the original cause. In other words, we may expect that during the very last stages, the black hole will emit gravitational waves with frequencies and rates of damping characteristic of itself, in the manner of a bell sounding its last dying pure tones.

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[2]Regarding the Schwarzschild solution, see Chandrasekhar and Detweiler (1975*a*), Gunter (1980*a*), Cunningham *et al.* (1978, 1979). Regarding the Kerr solution, see Detweiler (1977*a*, 1978, 1979, 1980) and Detweiler and Szedenits (1979).

These considerations underlie the formulation of the concept of the *quasinormal modes* of a black hole. In other words, these modes determine the *pure dying tones* of a perturbed black hole (Chandrasekhar, 1983).

Precisely, the quasinormal modes are defined by the solutions of the perturbation equations belonging to the complex characteristic frequencies and satisfying the boundary conditions appropriate for purely ingoing waves at the horizon and purely outgoing waves at infinity.

The study of the quasinormal modes of a black hole has attracted much attention in view of the detection of the gravitational radiation. It plays also a fundamental role in the study of the stability of black hole. [The stability of the Schwarzschild black hole is established (Regge and Wheeler, 1957).] On the other hand, the author professes, together with H.-J. Treder (personal communication), that there are "enormous hopes of Peterson's method of gravitational lens *interference* to detect black holes, brown dwarfs, super Jupiters, clear up the missing mass problem." Perhaps the gravitational quasinormal modes of black holes may be observable by advanced *laser-interferometric gravitational-wave detectors.*

## 2. SCALAR MASSIVE PERTURBATIONS OF BLACK HOLES

### 2.1. Wave Equation of Scalar Massive Perturbations

The scalar perturbations of black holes could be observationally relevant if "boson stars" prove to be viable candidates for dark matter and missing mass, respectively. If an object made up of self-gravitating scalar fields (Simone and Will, 1992) should become unstable and collapse to a black hole, it would radiate gravitational and scalar waves in the form of quasinormal modes. A purely spherical collapse will radiate scalar waves, but not gravitational waves. In general, this motivates the interest in *massive scalar quasinormal modes* (Simone and Will, 1992; Gal'tsov and Matiukhin, 1992).

These resonant, complex (Colpi *et al.*, 1986; Breit *et al.*, 1984; Gleiser and Watkins, 1989; Seidel and Suen, 1990; Friedberg *et al.*, 1987; Lee and Pang, 1989) or real (Seidel and Suen, 1991) quasinormal modes are characteristic of the Schwarzschild, Kerr, etc., geometries; for each kind of external perturbation (e.g., scalar, electromagnetic, gravitational) they have discrete, specific spectra (Simone and Will, 1992; Gal'tsov and Matiukhin, 1992). They are the solutions of a second-order hyperbolic partial differential equation (Teukolsky, 1973; Press and Teukolsky, 1973)

$$\mathscr{L}^{(s)}\Psi^{(s)} = T \tag{1}$$

where $\mathscr{L}^{(s)}$ is a second-order linear differential operator depending on $s$, the spin weight of the field (e.g., scalar, $s=0$; electromagnetic, $s=+1$; gravitational, $s=\pm 2$), and on the black hole parameters (e.g., mass $M$; angular momentum per unit mass $a$; charge $Q$), $\Psi^{(s)}$ is a variable depending on the field and $T$ is a function of the source of the field (in vacuum, $T=0$).

Mathematically, the normal modes correspond to solutions of equation (1) with a complex frequency

$$\Psi^{(s)} = \Phi \sim e^{-i\omega t}, \qquad w = w_r + i\omega_i \tag{2}$$

with $w_i < 0$ (decay of amplitude in time), and satisfying the boundary conditions of purely outgoing waves, or waves that propagate away from the potential barrier at $+\infty$ or $-\infty$, the latter corresponding to traveling across the horizon to the interior of the black hole.

In the case of massive scalar quasinormal modes, $\Psi^{(s)} = \Phi$ (for $s=0$) is given by the Klein–Gordon equation

$$(\Box - \mu^2)\Phi = 0 \tag{3}$$

where $\Box$ is the curved spacetime d'Alembertian on the black hole background, and $\mu$ is the inverse Compton wavelength characteristic of the field (Simone and Will, 1992; Gal'tsov and Matiukhin, 1992).

For Schwarzschild and Kerr black holes the massive scalar wave equation (3) in Boyer–Lindquist coordinates $r$, $\theta$, $\varphi$ and with *real frequency* $\omega$ (Boyer and Lindquist, 1967; Misner *et al.*, 1973) has the form

$$\left(\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right)\frac{\partial^2\Phi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\Phi}{\partial t\,\partial\varphi} + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\Phi}{\partial\varphi^2}$$
$$-\frac{\partial}{\partial r}\left(\Delta\frac{\partial\Phi}{\partial r}\right) - \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial\Phi}{\partial\theta}\right) - \mu^2\rho^2\Phi = 0 \tag{4}$$

Here, $M$ is the black hole mass, $aM$ its angular momentum $(0 \le a \le M)$, $\rho^2 = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 - 2Mr + a^2$ (Simone and Will, 1992; Gal'tsov and Matiukhin, 1992). The form of solution

$$\Phi(r,\theta,\varphi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty} g(\omega)R_{lm\omega}(r)S_{lm\omega}(\theta)e^{im\varphi}e^{-i\omega t}\,d\omega \tag{5}$$

leads to separation of variables in equation (4), resulting in ordinary differential equations for $R_{lm\omega}(r)$ and $S_{lm\omega}(\theta)$.

In equation (5), $l$ is the wave's angular momentum and $m$ its azimuthal projection. The separation of equation (4) into radial and angular variables gives (Teukolsky, 1973; Simone and Will, 1992; Gal'tsov and Matiukhin, 1992)

$$\Delta \frac{d}{dr}\left(\Delta \frac{dR_{lm\omega}}{dr}\right) + [\omega^2(r^2+a^2)^2 - 4Mam\omega r + m^2a^2]R_{lm\omega}$$
$$- (\omega^2 a^2 + \mu^2 r^2 + \lambda^{lm}_{(\omega\mu)})\Delta R_{lm\omega} = 0 \tag{6}$$

$$\frac{1}{\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dS_{lm\omega}}{d\theta}\right) + [a^2(\omega^2-\mu^2)\cos^2\theta - \frac{m^2}{\sin^2\theta} + \lambda^{lm}_{(\omega\mu)}]S_{lm\omega} = 0 \tag{7}$$

The solutions of equation (7), $S_{lm\omega}(\theta) - s$, are spheroidal (oblate) wavefunctions; in the Schwarzschild case, when $a = 0$, they reduce to the usual spherical harmonics

$$S_{lm\omega}(\theta)e^{-im\varphi} = S^m_l(-a^2c^2, \cos\theta) \underset{a=0}{\longrightarrow} Y^m_l(\cos\theta), \qquad c^2 = \omega^2 - \mu^2 \tag{8}$$

In equation (6) and (7) the eigenvalues $\lambda^{lm}_{(\omega\mu)}$ of the spheroidal wavefunctions (Flammer, 1957; Seidel, 1989) are given by an expansion in even powers of $ac$,

$$\begin{aligned}
\lambda^{lm}_{(\omega\mu)} &= \sum_k f^{lm}_{(2k)}\,(ac)^{2k} \\
f^{lm}_0 &= l(l+1) \\
f^{lm}_2 &= h(l+1, m) - h(l, m) - 1 \\
&\;\vdots \\
h(l, m) &= \frac{l(l-m)(l+m)}{2(l-\frac{1}{2})(l+\frac{1}{2})}
\end{aligned} \tag{9}$$

The remaining coefficients are similarly functions of $l$ and $h(l, m)$ (Seidel, 1989). Hence, $\lambda^{lm}_{(\omega\mu)}$ is invariant under both $\omega \to -\omega$ and $m \to -m$.

In general, equation (6) for $R_{lm\omega}(r)$ is not solvable analytically, but with the changes

$$Y_{lm\omega}(r) = (r^2+a^2)^{1/2}R_{lm\omega}(r), \qquad \frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta} \tag{10}$$

it becomes a Schrödinger-like radial equation or Regge–Wheeler-like equation

$$\frac{d^2Y_{lm\omega}}{dr^{*2}} + [\omega^2 - V(\omega, r)]Y_{lm\omega} = 0 \tag{11}$$

with a generalized Regge–Wheeler potential

$$V(\omega, r) = \frac{\Delta\mu^2}{r^2+a^2} + \frac{4Mam\omega r - a^2m^2 + \Delta(\lambda^{lm}_{(\omega\mu)} + (\omega^2-\mu^2)a^2)}{(r^2+a^2)^2}$$
$$+ \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2+a^2)^3} - \frac{3\Delta^2 r^2}{(r^2+a^2)^4} \tag{12}$$

In equations (10) and (11) $r^*$ is the "tortoise" coordinate that spans the interval $(-\infty, \infty)$ when the radial coordinate $r$ is in $(r_H, \infty)$, and $r_H$ is the horizon of the black hole (Misner *et al.*, 1973).

### 2.2. Solutions of Wave Equation of Scalar Massive Perturbations

The problem of solving equation (6) with boundary conditions has been studied in different ways. Since equation (6) is insolvable analytically, it is transformed into equation (11) by equation (10), and after this equation (11) is solved by approximations or polynomial methods or other techniques, and the polynomial or approximate or other solutions should be connected with the general solution $\Phi(r, \theta, \varphi, t)$ by equation (5). However, this does not always happen, as the following short summary shows. These methods of solution investigate the Regge–Wheeler-like equation or Schrödinger-like equation (Teukolsky, 1973) exclusively. [Naturally, equation (11) is also not solvable analytically, in general.]

One approach requires selecting a discrete value of $\omega$, integrating the differential equation, and checking that boundary conditions are satisfied (Chandrasekhar and Detweiler, 1975*b*; Detweiler, 1977*b*, 1980). Since this happens only for a discrete set of values, the complex $\omega$-plane has to be completely surveyed in the searching process. This can be expensive and time-consuming, especially if one is interested in variations of the results with a wide range of possible parameters.

Other techniques include the use of a parametrized approximation of the potential for which exact solutions are known (Blome and Mashhoon, 1984; Ferrari and Mashhoon, 1984; Gunter, 1980*b*, 1981), an analytical–numerical hybrid (Leaver, 1985) using an infinite-series representation of the solutions of the radial wave equation with numerical treatment of the quasinormal mode equation involving continued fractions, and the Laplace-transform method (Nollert and Schmidt, 1992).

Some authors use an alternative technique, a semianalytical approach that calculates the quasinormal mode frequencies using a WKB approximation (Schutz and Will, 1985; Iyer and Will, 1987; Iyer, 1987; Kokkotas and Schutz, 1988; Seidel and Iyer, 1990; Guinn *et al.*, 1990). It relies on the similarity between the black hole radial perturbation equation and the one-dimensional Schrödinger equation for a potential barrier [see the previous references, together with Outsuki and Futamase (1991), Leaver (1992), Gal'tsov and Matiukhin (1992), and Simone and Will (1992)].

However, in case of radial Schrödinger equations of quantum mechanics, the polynomial method cannot be used in the case of a continuous spectrum of energy $E$. (This case corresponds to the scattering processes from the viewpoint of quantum mechanics.) Furthermore, the quasiclassical

WKB approximation can be used for slowly changing $V(r)$ potentials only (Mészáros, 1989):

$$(m_e\hbar/p^3)\cdot|dV/dr|\ll 1 \tag{13}$$

In the case of the frequently occurring power potential $V(r)=\alpha r^{-\beta}$, $\beta>0$, equation (13) has the form

$$r\gg[(m_e\hbar\cdot|\alpha|\cdot\beta)/p^3]^{1/\beta+1}\neq 0 \tag{14}$$

It can be seen from equation (12) and inequality (14) that in the case of $V(r)=\alpha r^{-\beta}$, the solutions of (11) cannot be WKB approximated about $r=0$ (Mészáros, 1989) if $a=0$. Furthermore, it can be seen from equation (13) that in the case of $p\to 0$ the approximation is not valid (Mészáros, 1989). The approximation is also not valid at the turning point $E=V(r)$, so the WKB wavefunction produces the approximate solution of the Schrödinger-like equations only away from the turning points (Mészáros, 1989).

The approximate, polynomial, numerical, and other solutions are essential, but the determinations are difficult and depend on $V(r)$. Moreover, the radial standing wave solutions $R_{lm\omega}(r)$ of the original equation (6), more precisely, the radial structure of the black hole and its neighborhood with respect to problem of detection, has not been sufficiently investigated. In connection with equation (6), the question is raised: Can the general nature of the solutions of equation (6) be investigated without knowledge of the analytical, approximate, polynomial, numerical, or combined solutions with the help of a qualitative theory of second-order linear differential equations so the results are true with respect to classes of fast-changing or singular potentials?

## 3. QUALITATIVE INVESTIGATION OF EQ. (6)

Not enough use has been made of the results of the qualitative theory of the second-order linear equations for equation (6), notwithstanding that we have managed to point out on this basis many intriguing inconsistencies of axial and time-symmetric gravitational waves in the case of singular potentials (Mészáros, 1989). The application of the qualitative theory in cosmology has also resulted in many remarkable problems (Mészáros and Molnár, 1990, 1991).

Equation (6) is a second-order linear differential equation with the form

$$\left.\begin{array}{c}\Delta(\Delta R')'+Q(r)R=0\\ Q(r):=\omega^2\{(r^2+a^2)^2-a^2\Delta\}-4Mam\omega r+m^2a^2-\mu^2\Delta r^2-\lambda\Delta\\ \Delta:=r^2-2Mr+a^2\end{array}\right\} \tag{15}$$

Hereafter, the indices of $R(r)$, $S(\theta)$, and $\lambda$ are omitted.

The qualitative theory of second-order linear equations (Ledermann and Vajda, 1982) attempts to answer questions such as the following: Is a solution always bounded? Are there functions bounding a solution from above or from below? Is a solution monotone? Does a solution have a finite or infinite number of zeros? Is a solution stable? Equation (15) has the trivial solution $R(r) = 0$ for $0 \leq r < \infty$. This trivial solution will be excluded from further consideration.

### 3.1. The Case of $\Delta \neq 0$

If $\Delta \neq 0$, that is, if $r \neq r_\pm = M \pm (M^2 - a^2)^{1/2}$, then the radial equation (15) with the transformation

$$R(r) = P(r) \exp\left\{-\frac{1}{2}\int \frac{\Delta'}{\Delta}\, dr\right\}$$

leads to the Sturm–Liouville form (Mészáros, 1989):

$$\left.\begin{aligned} &P'' + q(r)P = 0 \\ q(r) :=& \left[\frac{\omega^2(r^2+a^2)^2 - 4Mam\omega r + m^2a^2 - \frac{3}{2}(2r-2M)^2}{\Delta^2}\right. \\ &\left. - \frac{(\omega^2a^2 + \mu^2r^2 + \lambda) - (1 + r - M)}{\Delta}\right] \end{aligned}\right\} \tag{16}$$

Because of this transformation $R(r) = P(r)\Delta^{-1/2}$, the number of zeros of solutions, that is, the oscillation, is not influenced by multiplying (or dividing) by $\Delta^{-1/2}$ and thus is the same for equations (15) and (16) (Mészáros, 1989).

In equation (16), $q(r)$ is factorizable as follows:

$$q(r) = q_1(r) + q_2(r) + q_3(r) + q_4(r) + q_5(r) + q_6(r) \tag{17}$$

where

$$\left.\begin{aligned} q_1(r) &:= \frac{\omega^2(r^4 + 2a^2r^2 + a^4)}{r^4 - 4Mr^3 + 4M^2r^2 + 2a^2r^2 - 4Ma^2r + a^4} \\ q_2(r) &:= \frac{-4Mam\omega r}{r^4 - 4Mr^3 + 4M^2r^2 + 2a^2r^2 - 4Ma^2r + a^4} \\ q_3(r) &:= \frac{m^2a^2}{r^4 - 4Mr^3 + 4M^2r^2 + 2a^2r^2 - 4Ma^2r + a^4} \\ q_4(r) &:= \frac{3(-4M^2 + 8Mr - 4r^2)2^{-1}}{r^4 - 4Mr^3 + 4M^2r^2 + 2a^2r^2 - 4Ma^2r + a^4} \\ q_5(r) &:= -\frac{\omega^2a^2 + \mu^2r^2 + \lambda}{r^2 - 2Mr + a^2} \\ q_6(r) &:= \frac{M - r - 1}{r^2 - 2Mr + a^2} \end{aligned}\right\} \tag{18}$$

If $r \to \infty$ (asymptotic case), then $q_1(r) \to \omega^2$, $q_2(r) \to 0$, $q_3(r) \to 0$, $q_4(r) \to 0$, $q_5(r) \to -\mu^2$, and $q_6(r) \to 0$. Namely, in the case of $r \to \infty$, $q(r) \to \omega^2 - \mu^2$, that is,

$$\lim_{r \to \infty} q(r) = \omega^2 - \mu^2 \tag{19}$$

It can be seen from equation (19) that at $r = \infty$ equation (16) has the form

$$P'' + (\omega^2 - \mu^2)P = 0 \tag{20}$$

Using the transformation $R(r) = P(r) \cdot \Delta^{-1/2}$, we obtain the solution of equation (20) at infinity

$$\text{(i)} \quad R(r) = A_1 \sin([\omega^2 - \mu^2]^{1/2} r + B_1) \cdot \Delta^{-1/2}$$
$$= \frac{A_1 \sin([\omega^2 - \mu^2]^{1/2} r + B_1)}{(r^2 - 2Mr + a^2)^{1/2}} \quad \text{if} \quad \omega > \mu \tag{21a}$$

$$\text{(ii)} \quad R(r) = (A_2 r + B_2) \cdot \Delta^{-1/2}$$
$$= \frac{A_2 r + B_2}{(r^2 - 2Mr + a^2)^{1/2}} \quad \text{if} \quad \omega = \mu \tag{22a}$$

$$\text{(iii)} \quad R(r) = [A_3 \exp(|\omega^2 - \mu^2|^{1/2} r)$$
$$+ B_3 \exp(-|\omega^2 - \mu^2|^{1/2} r)] \cdot \Delta^{-1/2} \quad \text{if} \quad 0 < \omega < \mu \tag{23a}$$

In the solutions $A_1$, $A_2$, and $A_3$ and $B_1$, $B_2$, and $B_3$ are arbitrary integration constants. Because of the correspondence of theory and observations, we obtain bounded solutions in equation (23a) if and only if $A_3 \equiv 0$.

Hence, in the case of $r \to \infty$ we expect solutions similar to the solutions (21a), (22a), and (23a). Applying the usual perturbation technique (Kato, 1966) for the asymptotic case, the more exact analysis gives that

$$\text{(i)} \quad R(r) \sim A_4 \sin\Big[(\omega^2 - \mu^2)^{1/2} r$$
$$- M(2\omega^2 - \mu^2) \log r + B_4 + O\left(\frac{1}{r}\right)\Big] \cdot \Delta^{-1/2} \quad \text{if} \quad \omega > \mu \tag{21b}$$

$$\text{(ii)} \quad R(r) \sim A_5 r^{-3/4} \sin[(8M\mu^2)^{1/2} r^{1/2} + B_5)] \quad \text{if} \quad \omega = \mu \tag{22b}$$

$$\text{(iii)} \quad R(r) \sim A_6 r^{M(2\omega^2 - \mu^2) - 1}$$
$$\times \exp(-|\omega^2 - \mu^2|^{1/2} r) \cdot \Delta^{-1/2} \quad \text{if} \quad 0 < \omega < \mu \tag{23b}$$

In the solutions (21b), (22b), and (23b), $A_4$, $A_5$, and $A_6$ and $B_4$, $B_5$, and $B_6$ are also arbitrary integration constants. Furthermore, in equation (21b), $|rO(1/r)|$ is bounded.

It can be seen from the solutions (21a)–(23a) and (21b)–(23b) that in the cases of $r = \infty$ and $r \to \infty$ the rotation of black holes cannot play a role. That is, the Schwarzschild black hole and Kerr black hole behave similarly at infinity (for the observer).

If $r \to 0$ (singular case), then it can be seen from equations (18) that $q_1(r) \to \omega^2$, $q_2(r) \to 0$, $q_3(r) \to m^2/a^2$, $q_4(r) \to -6M^2/a^2$, $q_5(r) \to -(\omega^2 a^2 + \lambda)/a^2$, and $q_6(r) \to (M-1)/a^2$. Namely, in the case of $r \to 0$, $q(r) \to (6M^2 + M + m^2 - \lambda - 1)/a^2$, that is,

$$\lim_{r \to 0} q(r) = \frac{-6M^2 + M + m^2 - \lambda - 1}{a^2} \qquad (a \neq 0) \tag{24}$$

It can be seen from equation (24) that for $r = 0$, equation (16) has the form

$$P'' + \left(\frac{-6M^2 + M + m^2 - \lambda - 1}{a^2}\right) P = 0 \tag{25}$$

Using the transformation $R(r) = P(r) \cdot \Delta^{-1/2}$, we obtain the solution at the singular point in the case of a Kerr black hole ($a \neq 0$),

(i) $R(r) = A_7 \sin(Cr + B_7) \cdot a^{-1}, \quad C := [(-6M^2 + M + m^2 - \lambda - 1)a^{-2}]^{1/2}$

$$\text{if} \quad -6M^2 + M + m^2 > \lambda + 1 \tag{26}$$

(ii) $R(r) = (A_8 r + B_8)a^{-1}, \qquad \text{if} \quad -6M^2 + M + m^2 = \lambda + 1 \qquad (27)$

(iii) $R(r) = [A_9 \exp(|C|^{1/2} r) + B_9 \exp(-|C|^{1/2} r)]a^{-1}$

$$\text{if} \quad -6M^2 + M + m^2 < \lambda + 1 \tag{28}$$

In the solutions (26)–(28), $A_7$, $A_8$, and $A_9$ and $B_7$, $B_8$, and $B_9$ are also arbitrary integration constants.

In the case of a Schwarzschild black hole ($a = 0$), if $r \to 0$, then equation (15) has the form

$$\Delta(\Delta R')' = 0 \tag{29}$$

Since $\Delta \neq 0$, only $\Delta \to 0$, so equation (29) becomes

$$\Delta R' = A_{10} \tag{30}$$

where $A_{10}$ is also an arbitrary integration constant. The solution of equation (30) is

$$R(r) = -\frac{A_{10}}{2M} \ln\left|\frac{r}{r - 2M}\right| + B_{10} \qquad (a = 0) \tag{31}$$

where $B_{10}$ is also an arbitrary integration constant.

In the nonasymptotic or nonsingular cases, we should apply the other results of the qualitative theory of second-order linear equations (Mészáros, 1989).

### 3.2. The Case of $\Delta = 0$

If $r = r_\pm = M \pm (M^2 - a^2)^{1/2}$, then $\Delta = 0$. In addition to the cases $r \to 0$ and $r \to \infty$, the investigation of this case is very important: How do the solutions behave in the neighborhood of event horizons $(r - r_\pm) \sim 0$, or rather, while crossing them?

Since now $\Delta = 0$, the method applied in Section 3.1 cannot be used. It can be seen from equation (15) that

$$Q(r_\pm) := (2\omega M r_\pm - ma)^2 \tag{32}$$

On the basis of equation (32), there exist two cases: $Q(r_\pm) > 0$ or $Q(r_\pm) = 0$.

1. The typical case is when $Q(r_\pm) > 0$. Combining the usual perturbation technique (Kato, 1966) with the local investigation, we find the solution

$$R(r) \sim A_{11} \sin\left(\frac{|2\omega M r_\pm - ma|}{(M^2 - a^2)^{1/2}} \log|r - r_\pm| + B_{11}\right) \tag{33}$$

where $A_{11}$ and $B_{11}$ are also arbitrary integration constants. It can be seen from the solution (33) that in the case of $\Delta = 0$ and $Q(r_\pm) > 0$ the solution of equation (15) has infinitely many zeros in the intervals $r_+ - \varepsilon < r \le r_+$, $r_+ \le r \le r_+ + \varepsilon$ $(\varepsilon > 0)$ and $r_- - \varepsilon < r \le r_-$, $r_- \le r < r_- + \varepsilon$. Thus, the standing wave solution cannot be continued analytically across the event horizons $r_+$ and $r_-$, since the solution (33) is singular. It can be seen also from (33) that this solution is valid for the Schwarzschild black hole and the Kerr black hole simultaneously.

2. The atypical case is when $Q(r_\pm) = 0$. Then it can be seen from (32) that

$$\omega_\pm := \frac{ma}{2Mr_\pm} \tag{34}$$

which means a *favored frequency*. In the case of a Schwarzschild-like black hole $a \to 0$, and so $\omega_+ \to 0$ and $\omega_- \to \infty$. In case of an extreme Kerr black hole $a \to M$, and so $\omega_+ \to \omega_-$.

Combining the usual perturbation technique (Kato, 1966) with the local investigation, we can write the solution of equation (15) in the form of a power series

$$R(r) = \sum_{n=0}^{\infty} (-1)^n \frac{(r - r_\pm)^n}{(n!)^2} \tag{35}$$

It can be seen from the solution (35) that in case of $\omega \approx \omega_{\pm}$ the solutions can cross over (can be continued analytically across) the event horizons of a Kerr black hole. Namely, the solution is free from all kinds of singularity.

## 4. SUMMARY

It can be seen from the investigations of Section 3 on the basis of equation (34) that the standing wave solutions of equation (15) can be continued from infinity to $r_-$ in case of a small angular momentum $a \to 0$ (Schwarzschild-like case). However, these low-frequency $\omega_+$ solutions cannot cross over (cannot continue analytically beyond) $r_-$, since, on the basis of equation (34), this is possible at only the high frequency $\omega_-$. Namely, the frequency should increase continuously in the interval $r_- < r < r_+$ as $r \to r_-$, or jump at $r_-$, and vice versa. Since in the case of $a \to 0, r_- \to 0$, these solutions can be continued to the singularity $r = 0$.

In case of great angular momentum $a \to M$ (extreme Kerr case), according to equation (34), $\omega_+ \to \omega_-$ (in the limit $\omega_+ = \omega_- = m/2M$), and so because of equation (5), the solutions of equation (15) can be continued analytically from infinity to the singularity $r = 0$ at these frequencies.

On the basis of equations (5) and (34) it can be seen that in the case of Kerr and Schwarzschild black holes there can exist continuous standing wave solutions in the total interval $0 < r < \infty$. This means that we can see into the black holes at these frequencies with gravitational-wave detectors, studying their radial structure inside the event horizon, too.

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